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Analytic solutions of the Fisher equation

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Abstract. We analyse the Fisher equation *via* the expanded Painlevé analysis approach. We obtain its singular property, auto-Bäcklund transformation and analytic solutions including some interesting heteroclinic and homoclinic solutions.

1. Introduction

Many phenomena arising in biology can be modelled by travelling waves which compose an important class of solutions, see [1–3]. Most of the methods we have used are not constructive. For example, the phase-plane analysis illustrates whether a certain trajectory exists and whether the solution is stable. Sometimes we are also interested in analytic solutions, but so far we lack effective methods of finding such solutions.

Recently we succeeded in using the expanded Painlevé analysis for carrier flow equations in semiconductor devices, and obtained its analytic solutions [7]. Here we take the Fisher equation as an example to demonstrate how to use this approach for biological problems. As we know, the Fisher model describes the wave of advance of an advantageous gene. Many authors have studied this equation from different points of view. Moreover, people have looked for heteroclinic and homoclinic solutions, but have not found the explicit expressions of these solutions. In this paper, we get heteroclinic and homoclinic solutions *via* Painlevé analysis.

2. Singular property and auto-Bäcklund transformation

The Fisher equation is

$$u_t = u_{xx} + au(1-u) \tag{2.1}$$

where the positive constant a is a measure of intensity of selection. For the convenience of analysis, we make the substitution

$$\nu = \tfrac{1}{6}u \tag{2.2}$$

$$\tau = 5t. \tag{2.3}$$

Then (2.1) changes into

$$5\nu_\tau = \nu_{xx} + a\nu(1-6\nu). \tag{2.4}$$

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The key point of Painlevé analysis is to demonstrate that the following 'ansatz'

$$\nu=\sum_{j=0}^{\infty} \nu_j \phi^{j-p} \tag{2.5}$$

is 'single-valued' about the solution movable singular manifold $\phi=0$; that is, p is a positive integer, recursion relationships for ν_j are self-consistent, and the ansatz (2.5) has enough free functions in the sense of Cauchy–Kowalevskia theorem [5].

By substituting (2.5) into (2.4), we have

$$p=2 \tag{2.6}$$

and

$$\begin{aligned}5[\nu_{j-2,\tau}+(j-3)\nu_{j-1}\phi_\tau]\\ &=\nu_{j-2,xx}+2(j-3)\nu_{j-1,x}\phi_x+(j-3)\nu_{j-1}\phi_{xx}+(j-3)(j-2)\nu_j\phi_x^2\\ &\quad+a\nu_{j-2}-6a\sum_{k=0}^{j}\nu_{j-k}\nu_k \qquad \forall j\geq 0\end{aligned} \tag{2.7}$$

where $\nu_j\equiv 0$ for $j<0$. For $j=0$, we have

$$\nu_0=\frac{1}{a}\phi_x^2. \tag{2.8}$$

Thus using (2.8), (2.7) turns into

$$\begin{aligned}(j-6)(j+1)\nu_j\phi_x^2\\ &=5[\nu_{j-2,\tau}+(j-3)\nu_{j-1}\phi_\tau]-\nu_{j-2,xx}-2(j-3)\nu_{j-1,x}\phi_x-(j-3)\nu_{j-1}\phi_{xx}\\ &\quad-a\nu_{j-2}+6a\sum_{k=1}^{j-1}\nu_{j-k}\nu_k \qquad j\geq 2.\end{aligned} \tag{2.9}$$

Clearly, the resonance points are $j=-1$ and $j=6$. The point -1 corresponds to the arbitrary singular manifold function ϕ, while the point 6 corresponds to the free function ν_6. For $j=1$, we obtain from (2.7) that

$$\nu_1=\frac{1}{a}(\phi_\tau-\phi_{xx}). \tag{2.10}$$

Generally, we can get all functions of ν_j through (2.9) but the distinctive part of Painlevé analysis is the truncation technique. Because of (2.6), we let $\nu_j=0, \forall_j\geq 3$. Then

$$\nu=\frac{\nu_0}{\phi^2}+\frac{\nu_1}{\phi}+\nu_2. \tag{2.11}$$

Hence (2.9) gives

$$\nu_2=\frac{1}{12}+\frac{1}{a}\left[-\frac{1}{4}\left(\frac{\phi_{xx}}{\phi_x}\right)^2+\frac{1}{3}\left(\frac{\phi_{xxx}}{\phi_x}\right)-\frac{\phi_{x\tau}}{\phi_x}+\frac{1}{2}\frac{\phi_{xx}\phi_\tau}{\phi_x^2}-\frac{1}{12}\left(\frac{\phi_\tau}{\phi_x}\right)^2\right] \tag{2.12}$$

$$\frac{\phi_{xx}-\phi_\tau}{\phi_x}\nu_2=\frac{1}{12}\left(\frac{\phi_{xx}-\phi_\tau}{\phi_x}\right)+\frac{1}{a}\left[-\frac{1}{2}\frac{\phi_{xx\tau}}{\phi_x}+\frac{5}{12}\frac{\phi_{\tau\tau}}{\phi_x}+\frac{1}{12}\frac{\phi_{xxxx}}{\phi_x}\right] \tag{2.13}$$

and

$$5\nu_{2,\tau}=\nu_{2,xx}+a\nu_2(1-6\nu_2). \tag{2.14}$$

Equation (2.14) is the same as (2.1). Thus (2.11) is an auto-Bäcklund transformation for the Fisher equation. We have from (2.8), (2.10) and (2.11) that

$$\nu = \frac{1}{a}\left[\frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2}\right] \ln \phi + \nu_2 \tag{2.15}$$

where ϕ satisfies (2.13), (2.14) with (2.12).

3. Analytic solutions

In this section we follow the idea of Conte [6] to deliver the analytic solutions of the Fisher equation systematically.

There are two elementary invariances under homographic transformation

$$H: \quad \phi \to \frac{C_1\phi + C_2}{C_3\phi + C_4} \qquad (C_1C_4 - C_2C_3 \neq 0). \tag{3.1}$$

They are the Schwarzian derivative

$$S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{3.2}$$

and dimension of velocity

$$C = -\frac{\phi_\tau}{\phi_x}. \tag{3.3}$$

Furthermore, we define

$$L = -\frac{\phi_{xx}}{2\phi_x}. \tag{3.4}$$

Since

$$L_x = -L^2 - \tfrac{1}{2}S \qquad L_\tau = -CL_x - LC_x + \tfrac{1}{2}C_{xx}$$

(2.12) and (2.13) read

$$\nu_2 = \frac{1}{12} + \frac{1}{a}\left(C_x - \frac{1}{12}C^2 + \frac{1}{3}S - CL + L^2\right) \tag{3.5}$$

and

$$(-2L + G)\nu_2 = -\frac{1}{6}L + \frac{1}{12}C + \frac{1}{a}\left(\frac{C_{xx} + CS}{2} + \frac{5}{12}(CC_x - C_\tau)\right.$$
$$\left. + \frac{1}{12}S_x - \left(2C_x + \frac{2}{3}S + \frac{5}{6}C^2\right)L + 3CL^2 - 2L^3\right). \tag{3.6}$$

By substituting (3.5) into (3.6) and (2.14), we have

$$C_\tau = \frac{1}{5}C_x - \frac{7}{5}CC_x + \frac{6}{5}C_{xx} + \frac{2}{5}SC + \frac{1}{5}C^3 \tag{3.7}$$

and

$$\frac{10}{3}C_{xxx}-\frac{31}{3}CC_{xx}+\frac{2}{3}S_{xx}-\frac{1}{3}S_xC+\frac{5}{3}SC_x+\frac{19}{6}C^2C_x-\frac{5}{6}C_x^2$$
$$=\frac{19}{6}SC^2+\frac{1}{8}C^4-\frac{1}{6}S^2+\frac{a^2}{24}. \tag{3.8}$$

According to the compatibility of S and C, we get another equation as follows:

$$S_\tau=-C_{xxx}-2C_xS-CS_x. \tag{3.9}$$

We can obtain the solutions of S and C through (3.7)-(3.9), and then get solution ν_2 by (3.2), (3.3) and (3.5). For the simplicity of analysis, we make the substitution

$$\phi_x=V^{-2}. \tag{3.10}$$

Therefore (3.2) and (3.3) turn into

$$V_{xx}+\frac{S}{2}V=0 \tag{3.11}$$

$$V_\tau+CV_x-\tfrac{1}{2}C_xV=0. \tag{3.12}$$

We now focus on the simplest case, i.e. when S and C are constants. Then (3.7)-(3.9) become

$$2SC+C^3=0 \tag{3.13}$$

$$\frac{19}{6}SC^2+\frac{1}{8}C^4-\frac{1}{6}S^2+\frac{a^2}{24}=0. \tag{3.14}$$

The general solutions of the above equations are

(i) $$S=-\frac{b^2}{2}\qquad C=\varepsilon b \tag{3.15}$$

(ii) $$S=-3b^2\qquad C=0 \tag{3.16}$$

(iii) $$S=\frac{b^2}{2}\qquad C=\varepsilon ib \tag{3.17}$$

and

(iv) $$S=3b^2\qquad C=0 \tag{3.18}$$

with $\varepsilon=\pm1$ and $a=6b^2$ since $a>0$. From (3.15), we have the general solution

$$V=A_1\exp\left(\frac{b}{2}\xi_1\right)+B_1\exp\left(-\frac{b}{2}\xi_1\right) \tag{3.19}$$

where A_1 and B_1 are arbitrary constants, and $\xi_1=x-\varepsilon b\tau$. Thus from (3.10)

$$\phi=\frac{E_1\exp\left(\frac{b}{2}\xi_1\right)+F_1\exp\left(-\frac{b}{2}\xi_1\right)}{A_1\exp\left(\frac{b}{2}\xi_1\right)+B_1\exp\left(-\frac{b}{2}\xi_1\right)} \tag{3.20}$$

where E_1 and F_1 are arbitrary constants provided that $E_1 B_1 - F_1 A_1 = 1/b$. So we have from (3.6) that

$$\nu_2^{(1)} = \frac{1}{12} - \frac{\varepsilon}{12}\left(\frac{A_1 \exp\left(\frac{b}{2}\xi_1\right) - B_1 \exp\left(-\frac{b}{2}\xi_1\right)}{A_1 \exp\left(\frac{b}{2}\xi_1\right) + B_1 \exp\left(-\frac{b}{2}\xi_1\right)}\right) - \frac{A_1 B_1}{6\left[A_1 \exp\left(\frac{b}{2}\xi_1\right) + B_1 \exp\left(-\frac{b}{2}\xi_1\right)\right]^2}. \tag{3.21}$$

Generally, we can get other solutions through the auto-Bäcklund transformation (2.15) [4]. But in this case, the solution obtained by (2.15) is just the same as the original solution.

By the transformations of (2.2) and (2.3), we have

$$U_2^{(1)} = \frac{1-\varepsilon}{2} + \frac{\varepsilon}{1 + \frac{A_1}{B_1}\exp(b\eta_1)} - \frac{A_1 B_1}{\left(A_1 \exp\left(\frac{b}{2}\eta_1\right) + B_1 \exp\left(-\frac{b}{2}\eta_1\right)\right)^2} \tag{3.22}$$

where $\eta_1 = x - \frac{5}{6}\sqrt{6a}\,\varepsilon t$. Similarly, (3.16) leads to the following stationary solution

$$U_2^{(2)} = 1 - \frac{6A_2 B_2}{\left[A_2 \exp\left(\frac{\sqrt{6}}{2}bx\right) + B_2 \exp\left(-\frac{\sqrt{6}}{2}bx\right)\right]^2} \tag{3.23}$$

where A_2 and B_2 are arbitrary constants. Furthermore from (3.18), we obtain the complex solutions

$$U_2^{(3)} = \frac{1+\varepsilon}{2} - \frac{\varepsilon}{1 + \frac{A_3}{B_3}\exp(ib\eta_2)} + \frac{A_3 B_3}{\left[A_3 \exp\left(\frac{bi}{2}\eta_2\right) + B_3 \exp\left(-\frac{bi}{2}\eta_2\right)\right]^2} \tag{3.24}$$

and

$$U_2^{(4)} = \frac{A_4 B_4}{\left[A_4 \exp\left(\frac{\sqrt{6}}{2}bix\right) + B_4 \exp\left(-\frac{\sqrt{6}}{2}bix\right)\right]^2} \tag{3.25}$$

where A_3, B_3, A_4 and B_4 are arbitrary constants and $\eta_2 = x - \varepsilon bit$. It is easy to verify that if $A_4^2 = B_4^2 = C_0$ in (3.25), then the complex solutions will degenerate into real solutions. For instance, for $A_4 = B_4$

$$U_2^{(4)} = \frac{C_0}{4}\sec^2 \frac{\sqrt{6}}{2}bx$$

while for $A_4 = -B_4$

$$U_2^{(4)} = -\frac{C_0}{4}\operatorname{Cosec}^2 \frac{\sqrt{6}}{2}bx.$$

We next consider the asymptotic behaviours of the solutions as $|x| \to \infty$. We find out for $\varepsilon = 1$

$$U_2^{(1)} \to 1 \qquad U_{2,\eta}^{(1)} \to 0 \qquad \text{as } x \to -\infty$$

$$U_2^{(1)} \to 0 \qquad U_{2,\eta}^{(1)} \to 0 \qquad \text{as } x \to +\infty$$

while for $\varepsilon = 1$

$$U_2^{(1)} \to 0 \qquad U_{2,\eta}^{(1)} \to 0 \qquad \text{as } x \to -\infty$$

$$U_2^{(1)} \to 1 \qquad U_{2,\eta}^{(1)} \to 0 \qquad \text{as } x \to +\infty.$$

Since $(0, 0)$ and $(1, 0)$ are the critical points of the Fisher equation [3], the above solutions are heteroclinic solutions. As for the solution (3.23), we have

$$U_1^{(2)} \to 1 \qquad U_{2,\eta}^{(2)} \to 0 \qquad \text{as } |x| \to \infty.$$

Thus (3.23) is a homoclinic solution.

Clearly, we can obtain other analytic solutions if we can find out other solutions of equations (3.7)-(3.9).

The technique in this paper has been used for other equations in biological mathematics, for example the Nagumo equation [8].

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